Recent Studies on Optimisation for Elastic Stability of Cylindrical and Conical Shells

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SUMMARY

Optimisation of cylindrical and conical shells for elastic stability usually leads to stiffened shells. The buckling behaviour of stiffened cylindrical shells under various loads and load combinations is discussed with special reference to eccentricity of stiffeners. Design implications of combined loads are considered and an optimisation procedure is proposed. Variation of stiffener cross-section is studied as an approach to optimisation. For conical shells, variation of stiffener spacing is investigated as another direction of improvement in structural efficiency. Orthotropic shells are also considered briefly. Finally, some of the results for closely spaced stiffeners are evaluated with discrete stiffener theory.

SYMBOLS

a, b	distance between rings and stringers for a cylindrical shell
a	distance of the top from the vertex, along a generator in a truncated conical shell, (*see* Fig. 1)
a_δ	$=(a_{0\delta}/x^\delta)$, distance between rings for a conical shell (*see* Fig. 1)
$a_{0\delta}$	defined by eqn. (12) when $x=1$
a_n, b_n	defined by eqns. (16) of ref. 25
A_n, B_n, C_n	coefficients of displacements
A_1, A_2	cross-sectional area of stringers and rings respectively
c, d	the width and height of rings in conical shells

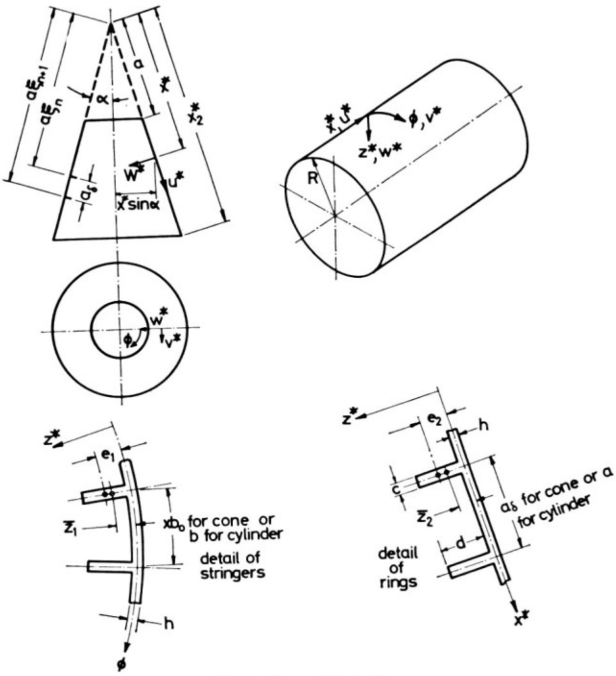

FIG. 1 — Notation

D $= Eh^3/12(1-\nu^2)$

$g(\psi)$ function of the taper ratio of a cone $\psi = 1-(R_1/R_2)$, given in ref. 12

h thickness of shell

$\bar{h}$ thickness of equivalent weight shell

I_{02} moment of inertia of a ring cross-section about the middle line of the sheet

K plate factor in plate buckling formula $N_{cr} = K(\pi^2 D/a^2)$

k_1, k_2, k_3, k_4 defined by eqns. (14) to (18)

l $= a(x_2 - 1)$ (*see* Fig. 1)

L length of shell between bulkheads

n number of half longitudinal waves

N_x critical axial load per unit circumference

p hydrostatic pressure

p^{out}, p^{in} critical pressure for outside and inside stiffeners

$c._{c.s.}, p_{v.s.}$ lateral pressure for uniform and non-uniform stiffening

P axial load

P^{out}, P^{in}	axial load for outside and inside stiffeners
$P_{c.s.}$, $P_{v.s.}$	axial load for uniform and non-uniform stiffening
r_a, r_p	defined by eqns. (4)
R, R_1, R_2	radius of cylindrical shell, and radii of small or large end of truncated cone respectively
t	number of circumferential waves
u, v, w	non-dimensional displacements, in cylinder $u=(u^*/R)$, $v=(v^*/R)$ and $w=(w^*/R)$ (*see* Fig. 1)
x^*, z^*, ϕ	axial co-ordinate along a generator, radial and circumferential co-ordinates
x	non-dimensional axial co-ordinate, $x=(x^*/a)$ for a conical shell, $x=(x^*/R)$ for a cylindrical shell
x_2	ratio of the distance of the bottom of a truncated cone from the vertex, to that of the top
$\bar{x}$	$=[(1+x_2)/2]$
Z	$=(1-\nu^2)^{1/2}(L/R)^2(R/h)$
$\bar{Z}$	$=Z/(I_{02}/ah^3)$
α	cone angle
β	$=(\pi R/L)$
γ	variation ratio, the ratio of weight of uniform part of stiffeners to total stiffener weight defined by eqn. (12)
$\zeta_1, \eta_{01}, \eta_{t1}, \mu_1, \chi_1$ $\zeta_2, \eta_{02}, \eta_{t2}, \mu_2, \chi_2$	changes in stiffnesses due to constant stringers and rings defined in refs. 13, 14, 18 and 25
$\zeta_1^*(x), \eta_{01}^*(x), \eta_{t1}^*(x), \mu_1^*(x_1), \chi_1^*(x)$ $\zeta_2^*(x), \eta_{02}^*(x), \eta_{t2}^*(x), \mu_2^*(x), \chi_2^*(x)$	changes in stiffnesses due to stringers and rings varying in the x direction
ξ_n	defined by eqn. (23)
$\eta_{2\delta}$	effective mean bending stiffness of the rings (eqn. 20)
λ	$=(PR/\pi D)$
λ_p	$=(R^3/D)p$
λ_r	defined by eqns. (4)
ν	Poisson's ratio
ρ_{av}	$=[(R_1+R_2)/2\cos\alpha]$, average radius of curvature for a truncated cone
$\bar{\rho}_{av}$	average radius of curvature for a sub-shell
ψ	$=1-(R_1/R_2)$, taper ratio

1. Introduction

Optimisation of the structural elements is a basic requirement in the design of aerospace vehicles. Extensive research efforts have therefore been directed not only towards refinement of structural analysis but also towards methods of minimum weight design, as is evident from Gerard's recent authoritative survey[1]. Buckling is the most probable mode of failure in aerospace structures and hence optimum design of aeronautical structures is mostly optimisation for stability. To-day this statement can also be extended to missiles and boosters. Whereas the shells of early missiles were pressure critical, advances in materials and technology have reinstated instability as the dominant criterion also in this branch of aerospace structures[2,3].

If one investigates the structural efficiency of thin shells, one finds that the prescribed typical dimensions for both aircraft fuselages and missiles usually eliminate monocoque shells as contenders for optimal configurations. This has been shown for cylindrical shells in bending[4], for cylindrical shells under axial compression and spherical caps under external pressure[2,3], for cylindrical shells under external pressure[5] and for conical shells under external pressure[6]. For optimisation one has, therefore, to turn to stiffened shells.

The structural optimisation of stiffened cylindrical shells has recently been studied by many investigators[7–11] and that of stiffened conical shells has also been investigated in detail[6]. These studies yielded important conclusions. In their aim to arrive at simple methods of optimisation, however, some investigators have made assumptions which cannot be fully justified. The authors of this paper believe that, although the final goal is minimum weight, a better understanding of the behaviour of stiffened shells under different loading conditions is needed to make optimisation procedures more reliable. Hence, the results of recent studies on the behaviour of stiffened shells are discussed before optimisation procedures are considered.

In aerospace structures, stiffened shells generally refer to shells stiffened geometrically with stringers and rings. Other practical methods of improving the structural efficiency of a shell in buckling are corrugated sheet, sandwich construction and orthotropic materials.

Sandwich construction, extensively studied by other authors[1] but not discussed in the present paper, shows great promise as an optimal configuration of shell structures[2], though the predicted weight saving potentials have not yet been fully realised. Directional variation of material properties as in orthotropic shells, can usually not compete with geometrical stiffening in optimisation for stability. But since non-structural design requirements sometimes demand monocoque configuration, orthotropic construction may be optimal. Furthermore, since orthotropic construction is often advantageous for internal pressure loading, possible weight savings under stability critical loadings are of interest and orthotropic shells are therefore discussed briefly. Corrugated shells are usually analysed as orthotropic or stiffened shells and only as such are they included here.

Now in stiffened shells — the major topic of the paper — the effectiveness of the stiffeners is determined by the following geometric parameters: spacing, shape, cross-sectional area and eccentricity.

The maximum distance between stiffeners is determined by the local buckling strength of the sub-shell between stiffeners. When the local conditions of all the sub-shells are similar, equal stiffener spacing is appropriate. But when the local conditions differ, unequal spacing may be preferable. For example, in a ring-stiffened conical shell under uniform external pressure(12), or in a ring-stiffened cylindrical shell under longitudinal varying axial loads (such as would be produced by the weight of the solid propellant in a rocket) non-uniform stiffener spacing is preferable. It should, however, be pointed out that the stiffener spacing discussed here, although optimum from the local instability point of view, is not necessarily the optimal spacing for minimum weight.

The shape of the stiffeners is very important since it determines their bending and torsional stiffness. The dominant influence of the bending stiffness is self-evident, but calculations for stiffened cylindrical shells covering a wide range of geometries(13,14) also brought out the importance of the torsional stiffness under axial compression and torsion.

The cross-section of the stiffeners is usually constant for the whole shell. But noticeable weight savings may be possible by variation of stiffener cross-section along the shell, as has already been pointed out in ref. 15. The effect of variation of stringer cross-section under axial compression and of variation of ring cross-section under hydrostatic pressure is therefore studied.

The eccentricity of the stiffeners may have a considerable effect on the buckling load of a shell. Although already noticed in 1947(16), the importance of the eccentricity effect has only lately been realised with the introduction of heavily stiffened shells in launching vehicles. Many investigators have recently studied this effect (*see* bibliography in ref. 14), and the authors have shown a physical explanation of the eccentricity effect under external pressure and axial compression(14,18). Outside stiffeners usually stiffen the shell more than inside ones, but there are many cases when the opposite is true, and for certain shell geometries an inversion of the eccentricity effect occurs. This behaviour has a noticeable influence on the optimisation of stiffened shells. Furthermore, for different loads, different eccentricity effects are observed and have to be included in the analysis of buckling under combined loads.

2. Uniformly Stiffened Cylindrical Shells

The buckling behaviour of ring-stiffened cylindrical shells under hydrostatic pressure and the current methods of analysis for buckling between rings and general instability are summarised in ref. 17. The general instability behaviour of stiffened shells under lateral and hydrostatic pressure is reconsidered in ref. 18 with emphasis on the influence of the eccentricity of stiffeners. The difference between buckling under lateral and hydrostatic pressure is stressed there, and it is pointed out that hydrostatic pressure is actually a particular case of combined loading with the corresponding possible buckling mode change. Whereas under lateral pressure, ring-stiffened shells always buckle with one longitudinal wave and many circumferential waves, under hydrostatic pressure a different buckle pattern with many longitudinal waves, which often is also axisymmetric, can appear. The transition from the $n=1$ mode to the $n \neq 1$ mode occurs when the axial stress component due to the hydrostatic pressure becomes dominant in short shells, or in shells with very stiff rings.

For both types of loading, rings are usually the most effective stiffeners. For hydrostatic pressure, however, a combination of rings and stringers may sometimes be more effective, when the stringers postpone the transition to the $n \neq 1$ mode. In short shells, outside rings yield higher critical pressures. As the shells become longer, or more precisely as the shell geometry parameter $Z=(1-\nu^2)^{1/2}(L^2/Rh)$ increases, an inversion of the eccentricity effect occurs and inside rings are stronger. For lateral pressure loading, the inversion of the eccentricity effect is practically independent of the geometry of the rings but depends very strongly on the shell geometry. For hydrostatic pressure, the inversion also depends primarily on the shell geometry but the bending stiffness of the rings affects it too. The reason for this difference in the eccentricity effect under lateral and hydrostatic pressure is the transition from the $n=1$ buckling mode to the $n \neq 1$ mode under the latter loading. This transition, which affects the inversion, depends on the moment of inertia of the rings, I_{02}. Hence if the geometry of the shell is represented by the Batdorf parameter Z, a well defined 'range of inversion' which spreads between $100<Z<500$, is found for lateral pressure loading. For hydrostatic pressure, a similar range of inversion can be found when a modified shell parameter $\bar{Z}=Z/(I_{02}/ah^3)$, which accounts for the dominant ring property, is employed instead of Z. In Fig. 2 (reproduced from ref. 18), the ratio of $(p^{\text{out}}/p^{\text{in}})$, where p^{in} is the critical pressure for inside stiffeners and p^{out} that for outside ones, is plotted against $\bar{Z}$. The discontinuities in the curves indicate transition from the $n=1$ buckling mode to the $n \neq 1$ mode. The range of inversion is well defined. When $\bar{Z}<35$ outside rings are more efficient whereas for $\bar{Z} > 65$ shells with inside rings are stronger. For lateral pressure, the variation of (p^{out}/p^{in}) with Z is given in ref. 18. For stringer-stiffened shells, no inversion of eccentricity effect occurs under lateral loading, and under hydrostatic pressure an inversion may appear only in extremely short shells.

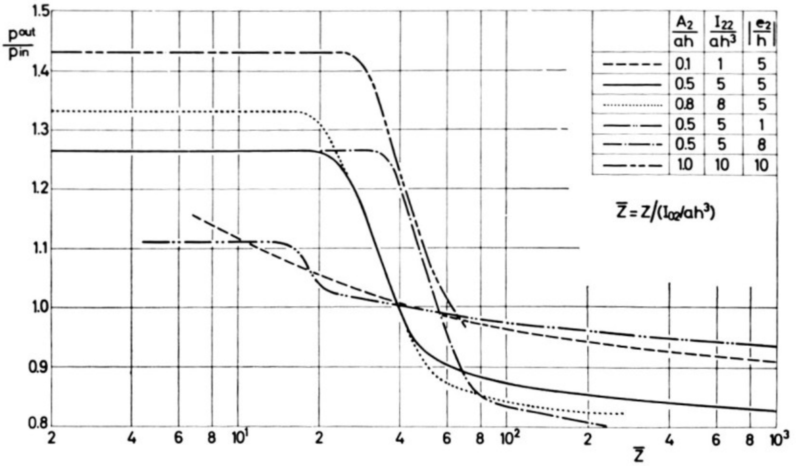

FIG. 2 — Variation of eccentricity effect with modified shell parameter $\bar{Z}$ for hydrostatic pressure loading. (Reproduced from Ref. 18)

Hence in practice, outside stringers are always better than inside ones for both types of loading.

Under axial compression, stiffeners play an even more important role in stabilising cylindrical shells. Unstiffened cylinders buckle at axial loads much below those predicted by classical linear theory. These very large discrepancies between experimental and theoretical buckling loads are attributed primarily to the post-buckling behaviour of axially compressed cylindrical shells and their imperfection sensitivity, and to a lesser extent to the influence of their boundary conditions (*see* for example ref. 19 for an up to date account). Closely stiffened cylinders under axial compression, on the other hand, show good agreement between experiment and linear theory(20,21,14).

The main stability contribution of stiffeners in this case is, therefore, the raising of the buckling load to the classical one, and the improvement of the structural efficiency within the framework of the behaviour predicted by linear theory, as in the case of external pressure, represents an additional gain. The results of ref. 14 show that even within linear theory stiffened shells are usually more efficient than unstiffened ones. Stringers are in general much more effective than rings in this function. Actually, a combination of stringers and rings is optimal as is shown in the calculations of the present paper as well as in refs. 9 and 11. The eccentricity effect is very pronounced in axially compressed stringer-stiffened cylinders and spectacular results have been observed also in tests[22]. Again, the eccentricity effect depends very strongly on the geometry of the shell, represented by the Batdorf parameter, while the geometry of the stringers only influences its magnitude. In Fig. 3 (reproduced

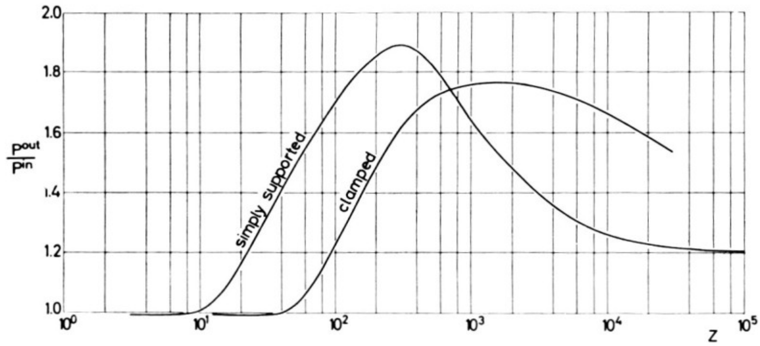

FIG. 3 — Variation of eccentricity effect under axial compression with shell geometry and boundary conditions. (Reproduced from Ref. 14)

from ref. 14) (P^{out}/P^{in}) is plotted against Z for simply-supported and clamped shells. An inversion of the eccentricity effect can again be observed, but here it occurs for extremely short shells. For all practical geometries, outside stringers yield higher buckling loads than inside ones. The designer should note that the eccentricity effect has a pronounced maximum which occurs for values of Z which are common in aerospace practice. This maximum and the shape of the curve are the result of the interplay of the two opposing contributions which make up the total eccentricity effect, discussed in detail in ref. 14. Figure 3 also shows that the eccentricity effect is approximately similar for clamped ends and simple supports (only two classical sets of boundary conditions out of the 8 possible supports are considered) and differs mainly in the position of the maximum and its magnitude.

In ring-stiffened cylinders the eccentricity reduces the buckling load with inside rings, whereas with outside rings the axisymmetric pattern, that is not influenced by eccentricity, dominates.

A recent study of the imperfection sensitivity of shells[23], maintains that some of the advantages of outside stringers predicted by linear theory, may not be materialised because of increased imperfection sensitivity that appears in shells with outside stringers over a substantial range of Z. This would mean that the additional gain by placing stringers outside is offset by a loss of not achieving linear theory buckling loads that one can usually expect for closely stiffened shells. Experiments so far[21] do not seem to support this fear. Moreover, the addition of rings to the stringer-stiffened shell under axial compression, advocated also by efficiency considerations of linear theory, should improve the situation due to the more stable post-buckling behaviour in the presence of rings[21,24].

The authors, therefore, feel that for the closely spaced stiffeners that are needed for minimum weight configurations or optimal designs in their vicinity, linear theory should adequately predict the buckling loads, and the assumption of identical imperfection sensitivity for stiffened and unstiffened cylindrical shells under axial compression employed in refs. 9 and 11 appears too conservative.

In order to obtain a realistic evaluation of the total effectiveness of stringers as stiffeners of axially compressed cylinders, one should, therefore, apply an empirical correction factor to unstiffened shells and use linear theory for the stiffened ones. This was done in ref. 14 and the 'corrected' structural efficiency of the stringer-stiffened shell is much higher than the uncorrected one that uses linear theory throughout. It is found that outside stringers are always more effective than equivalent thickening of the shell, irrespective of shell geometry, whereas inside stringers may be inferior to equivalent thickening for certain shell geometries.

The general instability of stiffened cylindrical shells under torsion is considered in detail in ref. 13. Rings are more efficient than stringers except for short shells. Large eccentricity effects are found for rings, and an inversion of the eccentricity effect occurs in the range of $1000 < Z < 4000$.

3. Combined Axial Compression and External or Internal Pressure

During the mission of a launch vehicle or missile it is subjected to combinations of axial and pressure loads. The general instability behaviour of stiffened cylindrical shells under combined loads is therefore studied. The analysis employs linear Donnell type equations and is an extension of that given in refs. 25 and 14. For classical simple supports, the third stability equation, eqn. (18) of ref. 25, becomes for axial compression and external or internal pressure,

$$\begin{aligned}\zeta_1(-n^3\beta^3 a_n)+\zeta_2(-2t^2-b_n t^3)+(1+\eta_{01})n^4\beta^4+(2+\eta_{t1}+\eta_{t2})n^2\beta^2 t^2\\ +(1+\eta_{02})t^4+12(R/h)^2[(1+\mu_2)(1+b_n t)+\nu n\beta a_n]-\lambda(n^2\beta^2/2)\\ -\lambda_p[(n^2\beta^2/2)+t^2]=0 \qquad (1)\end{aligned}$$

where $\lambda=(PR/\pi D)$ and $\lambda_p=(R^3/D)p$ (2)

n is the number of axial half waves, t the number of circumferential waves, μ_1, μ_2, η_{01}, η_{02}, η_{t1} and η_{t2} are the changes in stiffnesses due to stringers and rings, χ_1, χ_2, ζ_1, and ζ_2 are the changes in stiffnesses caused by the eccentricity of the stringers and rings, as in ref. 25, and a_n and b_n are given by eqns. (16) of ref. 25. When one of the load parameters, say λ_p is given, the second, say λ, is calculated from eqn. (1). Note that in eqn. (2) positive p represents external pressure and negative p internal pressure.

Computations have been made for many typical shells covering a wide range of shell and stiffener geometries. The details are given in section 3 of ref. 26. Here the relative efficiency of stringers and rings and their position is investigated for two typical shells, with $(L/R)=2{\cdot}0$ and $0{\cdot}5$, $(R/h)=1000$ and a stiffener weight ratio (ratio of total weight of the stiffened shell to that of the unstiffened shell) $(\bar{h}/h)=1{\cdot}5$ and $2{\cdot}0$, where $\bar{h}$ is the equivalent thickness of the stiffened shell and h is the wall thickness of the unstiffened shell

$$(\bar{h}/h) = [1+(A_1/bh)+(A_2/ah)] \tag{3}$$

The interaction curves for combined axial compression and external or internal hydrostatic pressure consist essentially of two straight lines that represent two different buckling modes, one with one longitudinal half wave, $n=1$, and one with many longitudinal waves, $n\neq 1$. Unstiffened cylindrical shells under the same combined load exhibit a similar behaviour. There the transition from the $n=1$ mode to the $n\neq 1$ mode occurs very near the zero pressure axis (it is sometimes assumed that this transition occurs exactly at the zero pressure axis, whereas actually it occurs at a small positive pressure — *see* ref. 27 — but still very near the zero pressure axis). In stiffened cylindrical shells, on the other hand, the transition appears at different places along the pressure axis depending on the stiffener geometry (*see* Fig. 4). Hence the interaction curves for stiffened and unstiffened shells differ considerably in shape and nature, and one cannot assume that the same interaction applies to both types of shells, as in ref. 10.

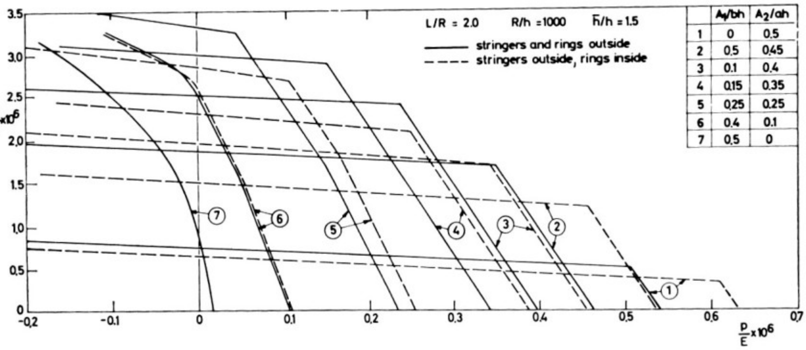

FIG. 4 — Interaction curves for different fractions of stiffener area allocated to rings and stringers

In Fig. 4 the weight ratio $(\bar{h}/h)=1{\cdot}5$ is kept constant and interaction curves are shown with different fractions of the stiffener area allocated to rings and stringers. The most effective distribution of stiffener material for uniformly spaced and constant area rings and stringers can be found from Fig. 4 for any

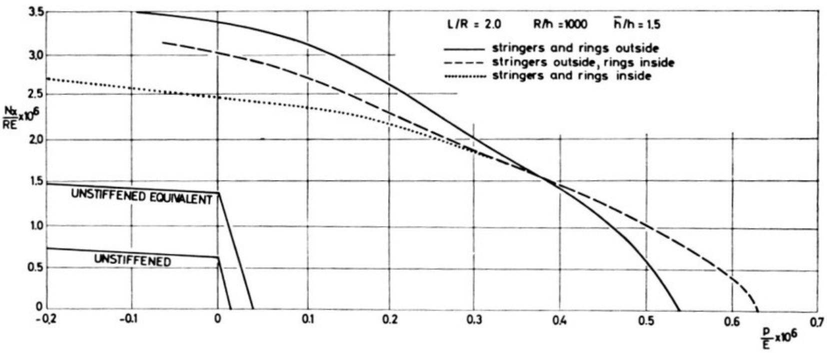

FIG. 5 — Interaction curves for most efficient distribution of stiffener material

combination of axial load and pressure. There is an interplay between the stiffening contribution of stringers and rings. The longitudinal stiffening of stringers postpones the $n \neq 1$ buckling mode. Since higher critical axial loads correspond to the $n=1$ mode than to the $n \neq 1$ mode, the interaction curve is raised, or in other words for a certain pressure a higher axial buckling load is attained. On the other hand, since increase in stringer area decreases that of the rings, and therefore the resistance to lateral pressure is reduced, the interaction curve shifts to the left. Along the pressure axis, the conclusions of ref. 18 that rings are the most effective stiffeners under hydrostatic pressure is reconfirmed, and along the axial compression axis a combination of about half the stiffener area allocated to rings and half to stringers is found to be most effective. (A similar conclusion is arrived at in ref. 9.)

It should be recalled here that the superiority of rings alone for stiffening against hydrostatic pressure does not always hold. As discussed earlier, two modes appear in buckling under hydrostatic pressure. Hence for certain values of Z, for which the $n \neq 1$ buckling mode would appear with rings only, the addition of stringers of very small area may suffice to cause transition to the $n=1$ mode and result in considerable increase in buckling pressure. For example, in Fig. 6, allocation of $2{\cdot}5\%$ of the total weight to stringers (inside rings and outside stringers) raises the buckling pressure by 47% and in Fig. 7 allocation of $1{\cdot}3\%$ of the total weight to stringers (rings inside and stringers outside) raises the critical pressure by 42%. It may be recommended therefore, that, when the modified stiffened shell parameter $\bar{\bar{Z}}<65$, stringers be added

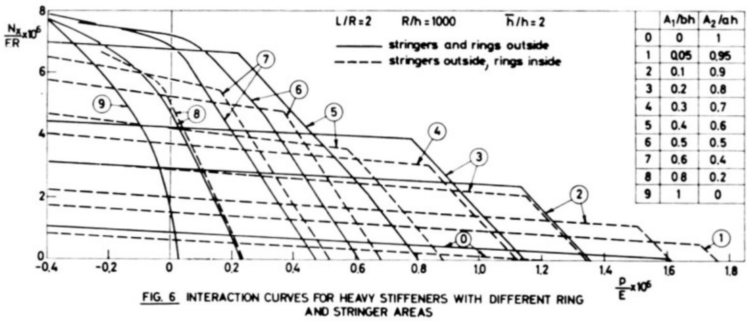

FIG. 6 — Interaction curves for heavy stiffeners with different ring and stringer areas

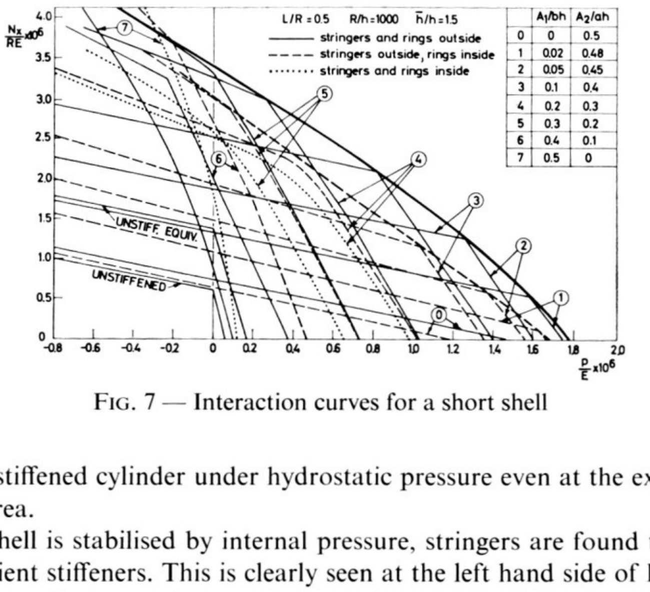

FIG. 7 — Interaction curves for a short shell

to a ring stiffened cylinder under hydrostatic pressure even at the expense of the ring area.

If the shell is stabilised by internal pressure, stringers are found to be the most efficient stiffeners. This is clearly seen at the left hand side of Figs. 4, 6 and 7, where the interaction curves for stringers only rise very rapidly with internal pressure and exceed those for stringers and rings. This is not surprising, since the internal pressure stabilises the shell mainly in the circumferential direction, and hence additional longitudinal stiffening is more important.

The influence of the position of the stiffener on the interaction curves is shown in Figs. 5 and 7. The curves shown are envelopes of the interaction curves for different weight distributions between stringers and rings for a constant stiffener weight ratio $(\bar{h}/h) = 1{\cdot}5$. These envelopes represent the maximum axial buckling load that can be attained with a given weight of stiffened shell for any hydrostatic pressure below the critical. The most efficient configuration for most of the range of combined loads is that with both stringers and rings on the outside. This could be expected from the behaviour of a stiffened shell under separate loads[14,18]. Stringers are the main stiffeners against the axial load component, and outside stringers are more effective than inside ones over the entire practical geometry range. For rings, on the other hand, which are the main stiffeners against the lateral load component, outside rings are more effective only in shells with small Z, and the eccentricity effect inverts as Z increases. Hence the conclusion that shells with rings and stringers on the outside are most efficient, holds for the entire range of combined loads only in short shells (*see* for example Fig. 7 where $\bar{Z}=13{\cdot}6$), whereas for long shells, inside rings and outside stringers are more efficient at the pressure end of the interaction curves.

It should be pointed out that the eccentricity effects for combined stiffening or combined loads are smaller than those corresponding to one type of stiffener only and separate loads. For example, in Fig. 7 at the axial compression axis, $(P^{\text{out}}/P^{\text{in}})$ for stringers only is about 1·38, whereas for combined stringers and rings of equal area $(P^{\text{out}}/P^{\text{in}})$ is 1·28. Or at about the middle of the interaction curve, at $(p/E)=0{\cdot}8\times10^{-6}$, $(P^{\text{out}}/P^{\text{in}})$ is 1·55 for $(A_2/ah)=0{\cdot}4$ and $(A_1/bh)=0{\cdot}1$. This reduction in eccentricity effect is due to the presence of both rings and stringers, whereas only either rings or stringers — depending on the dominant load — are directly influenced by the eccentricity effect.

The structural efficiency of stiffening is indicated in Figs. 5 and 7, by a comparison with equivalently thickened shells. The very large increases in buckling loads attained by stiffening, re-emphasise the relative inefficiency of monocoque shells. The fact that buckling loads for monocoque shells often fall much below the prediction of the linear theory considered here, whereas stiffened shells usually carry the 'linear' loads, discredits the monocoque shell even further.

4. Optimisation Procedure for Combined Loading

The analysis of the previous section lead to the conclusion that under combined axial compression and hydrostatic pressure some combination of stringers and rings represents the most effective stiffening against buckling. An optimisation procedure for combined loading is now proposed.

If one considers general instability, a typical interaction curve for a certain combination of rings and stringers is represented by the dotted line in Fig. 8(*a*). Point A represents the buckling mode transition and indicates the ratio of axial compression to hydrostatic pressure for which this stiffener configuration is most efficient. The part of the curve to the left of A can be approximated, with a slight conservative error, by a horizontal line. Considering local buckling between rings, one can draw some general conclusions about it from

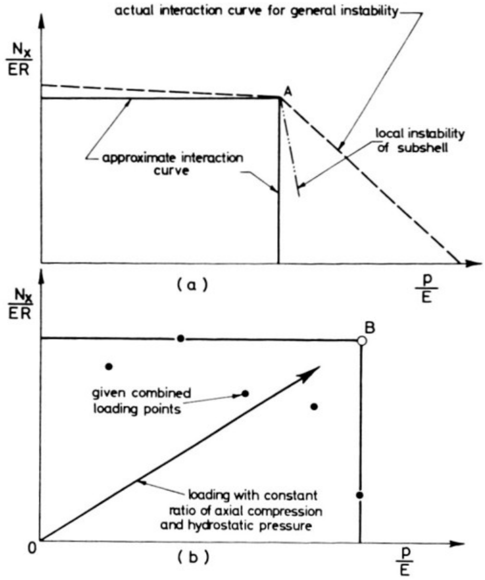

FIG. 8 — Optimisation procedure for combined loading

interaction curves like Figs. 4, 6 and 7. Local instability of sub-shells corresponds to buckling of shells stiffened by stringers only, or of unstiffened shells, which are observed to have interaction curves with the steepest slopes. Hence, if the part of the interaction curve in Fig. 8(*a*) to the right of *A* is approximated by a vertical line from *A*, this conservative approximation includes also the worst shape of interaction curve from the point of view of local buckling between rings. Local instability between stringers is essentially buckling of an unstiffened panel under combined load, for which the proposed approximate interaction curve is conservative.

Now consider a set of given ratios of axial compression and hydrostatic pressure as may occur in a typical mission of a vehicle. With the simple approximate interaction curve arrived at in Fig. 8(*a*), a single point *B* can immediately be found that covers all the actual combined loading conditions (Fig. 8(*b*)). Since the approximate interaction curve is conservative throughout, one is certain that the configuration that carried the load combination at *B* will be at least strong enough for all the actual combined loads. Hence the minimum or optimum weight problem of many combinations of axial compression and hydrostatic pressure is reduced to that of a single ratio of loads.

A minimum or optimum weight analysis similar to that given in refs. 9 and 11 for separate loads, can now be done for the load ratio represented by point *B* in Fig. 8(*b*). The load terms in eqn. (1) are changed to represent the manner in which point *B* of Fig. 8(*b*) is reached by loading with a constant load ratio. A new load parameter λ_r is introduced in eqn. (1) defined by

$$\lambda = r_a \lambda_r \quad \text{and} \quad \lambda_p = r_p \lambda_r \tag{4}$$

where r_a and r_p are coefficients representing the given ratio of λ and λ_p. Equation (1) becomes

$$\begin{aligned} &\zeta_1(-n^3\beta^3 a_n)+\zeta_2(-2t^2-b_n t^3)+(1+\eta_{01})n^4\beta^4+(1+\eta_{02})t^4 \\ &\quad +(2+\eta_{t1}+\eta_{t2})n^2\beta^2 t^2+12(R/h)^2[(1+\mu_2)(1+b_n t)+\nu n\beta a_n] \\ &\quad -\lambda_r\{[(r_a+r_p)/2]n^2\beta^2+r_p t^2\}=0 \end{aligned} \tag{5}$$

The limiting cases of separate loads are given by the following values of the coefficients:

$r_a=1, \quad r_p=0$ represents axial compression,

$r_a=0, \quad r_p=1$ represents external hydrostatic pressure,

$r_a=-1, \quad r_p=1$ represents external lateral pressure.

In combined loading, it is convenient to adjust the definition of the coefficients according to the dominant load.

When the axial compression P is dominant

$$r_a = 1 \quad \text{and} \quad r_p = (\lambda_p/\lambda) = (p\pi R^2/P) \tag{6}$$

and then

$$\lambda = \lambda_r \quad \text{and} \quad \lambda_p = (p\pi R^2/P)\lambda_r \tag{7}$$

Or, when the hydrostatic pressure is dominant,

$$r_a = (\lambda/\lambda_p) = (P/p\pi R^2) \quad \text{and} \quad r_p = 1 \tag{8}$$

and then

$$\lambda = (P/p\pi R^2)\lambda_r \quad \text{and} \quad \lambda_p = \lambda_r \tag{9}$$

With these definitions the optimisation procedure for the combined load B is identical to that of a given separate load, since the definitions ensure loading along the line OB, and the methods of refs. 9 and 11 can be directly applied. It should be pointed out, that on account of the conservatism of the approximate interaction curve, the resultant configuration is slightly heavier than the optimal. Furthermore, the difference in the applied stress for local instability calculations under axial compression and hydrostatic pressure, discussed below in connection with the optimisation of conical shells, complicates the local buckling calculations. The procedure, however, makes optimisation at all feasible, since it yields a single criterion for a variety of load ratios that actually have interaction curves of different shapes.

5. Variation of Stiffener Cross-Section

General instability in uniformly stiffened cylindrical shells under external pressure, axial compression or torsion usually occurs with one half-wave in the axial direction[13,14,18], notable exceptions being ring-stiffened cylinders under axial compression and short ring-stiffened cylinders under hydrostatic pressure. With the usual $n=1$ pattern, a logical approach to increase in structural efficiency is variation of stiffener rigidity along the shell in accordance with the buckling deflection. Since local stiffener instability is rarely a design criterion, the reduction in stiffener buckling strength is not considered. It is also assumed that the influence of stiffener rigidity on local shell instability is small (and for the case of hydrostatic pressure this assumption is justified in the next section) and therefore the change in this small effect may be disregarded. A detailed study of the general instability of simply-supported cylindrical shells with non-uniform stringers under axial compression and with non-uniform rings under hydrostatic pressure is given in ref. 28. The analysis uses the same model of a shell with 'distributed' stiffeners as refs. 25, 14 and 18. The constant changes in stiffnesses μ_1 to ζ_2 are replaced by 'starred' quantities $\mu_1^*(x)$ to $\zeta_2^*(x)$ which are continuous functions of x that have a maximum at the shell mid-length.

As a solution of the three stability equations (identical with eqns. (12) of ref. 25 except for the different stiffness terms which are functions of x) displacement series of the form

$$
\begin{aligned}
u &= \sin t\phi \sum_{n=1}^{\infty} A_n \cos n\beta x \\
v &= \cos t\phi \sum_{n=1}^{\infty} B_n \sin n\beta x \\
w &= \sin t\phi \sum_{n=1}^{\infty} C_n \sin n\beta x
\end{aligned}
\tag{10}
$$

are assumed. Each term of eqns. (10) satisfies the classical simple-support boundary conditions. In the general case of non-uniform stringers and rings none of the three stability equations are actually solved by eqns. (10). Hence two alternative methods of approximate solution are used: a straightforward Galerkin method and a method involving 'correcting coefficients' and minimisation of 'error loads' used in ref. 29 for stiffened conical shells. The two methods are discussed and compared in ref. 28.

The practical aim of the study is an evaluation of the possible weight savings by variation of stiffener area. If one considers stiffeners of rectangular cross-sections, the non-uniformity can be achieved by variation of stiffener height or width. Though height variation is clearly more effective, because stiffener moment of inertia is changed to the third power and the eccentricity is also affected, whereas width variation only changes the moment of inertia linearly and does not affect the eccentricity, the latter form of area variation may sometimes be the only practical possibility. Hence both forms of variation are studied.

For stringer-stiffened shells under axial compression two examples of height variation are considered: sinusoidal and linear (*see* Fig. 9). As the radial displacement of the uniformly stiffened shell varies sinusoidally, sinusoidal variation seems promising, whereas linear variation is obviously preferable for production. A mathematical analysis shows, that from the point of view of

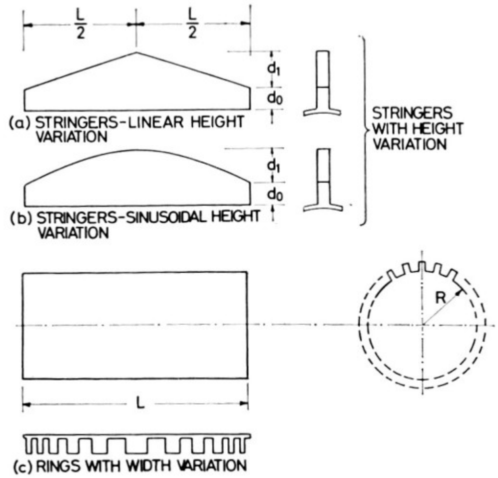

FIG. 9 — Types of stiffener area variations

general instability alone (in the single half-wave mode), concentration of the entire stringer area at mid-shell would be most efficient. Considerations of local buckling and of a possible two half-wave general instability mode, however, eliminate any practical significance of this result. Furthermore, considerations of local instability eliminate 'pure' variations of cross-section that would yield zero stringer area at the ends of the shell. Hence the varying part of the stringer is supplemented by a uniform part.

In Fig. 10 the ratio of critical axial load for stringers with sinusoidally varying cross-sectional area to that for equal weight uniform stringers ($P_{v.s.}/P_{c.s.}$) is plotted against Z. The stringers considered are fairly heavy, equivalent to uniform stringers with $(A_1/bh)=0{\cdot}5$ and $(e_1/h)=\pm 10$, as used in large boosters. The gains in critical load obtained by variation of stringer area are in general larger for shorter shells. Gains up to 33% are seen in Fig. 10. Different 'variation ratios' γ (the ratios of the weight of the uniform part of the stiffeners to the total stiffener weight) are plotted. $\gamma=0$ (no uniform part) is clearly an inefficient configuration, since the excessive weakening of the stringers causes transition from a single half-wave buckling mode to a many half-wave mode. The shell will then buckle in a number of bays, and as the bays near the ends are now 'understiffened', the buckling loads are

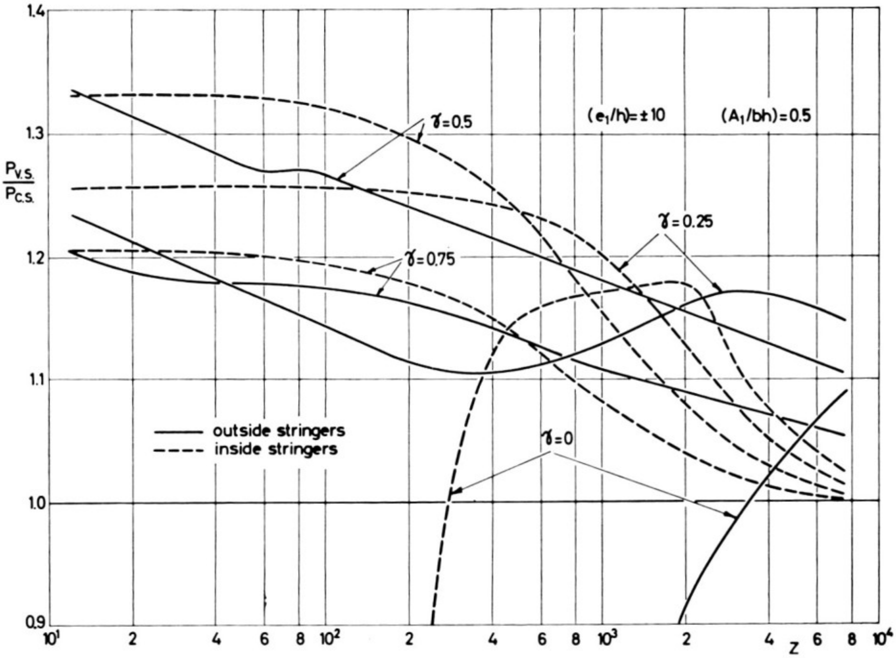

FIG. 10 — Influence of sinusoidal height variation of stringers on buckling load (axial compression)

lower with $\gamma=0$. The $(P_{v.s.}/P_{c.s.})$ ratio may even drop considerably below unity, as is seen in Fig. 10. $\gamma=0{\cdot}25$ yields much better results and $\gamma=0{\cdot}5$ appears to be roughly optimal, except for very large Z. Increases of 20–30% in buckling loads, can be achieved in practical configurations.

The decrease in $(P_{v.s.}/P_{c.s.})$ at large Z is the result of the reduction in structural efficiency of all stringers in long shells (discussed in ref. 14, *see* Figs. 6 and 11 there). The influence of the position of the stringers on $(P_{v.s.}/P_{c.s.})$ should be noted. In short shells inside stringers yield higher $(P_{v.s.}/P_{c.s.})$ ratios than outside ones, whereas in long shells larger gains are observed with outside stringers.

Linear and sinusoidal height variations are compared in Fig. 11, for outside stringers. With the approximately optimal variation ratio $\gamma=0{\cdot}5$, linear height variation approaches the efficiency of sinusoidal variation, and for $\gamma=0{\cdot}75$ linear height variation is even slightly more efficient. Since linear height variation presents less manufacturing problems than other forms, the 15–25% gain in buckling load seems promising. Some additional weight savings may be achieved by optimisation of the variation ratio γ[28], and a comparison between linear and sinusoidal height variations indicates that the optimal γ for linear variation is slightly larger than that for a sinusoidal one.

In Fig. 12 the effect of eccentricity on the $(P_{v.s.}/P_{c.s.})$ ratio is investigated for sinusoidal height variation. The eccentricity of the equivalent uniform stringer is varied for constant weight $(A_1/bh)=0{\cdot}5$ and constant $\gamma=0{\cdot}5$. Increase in the

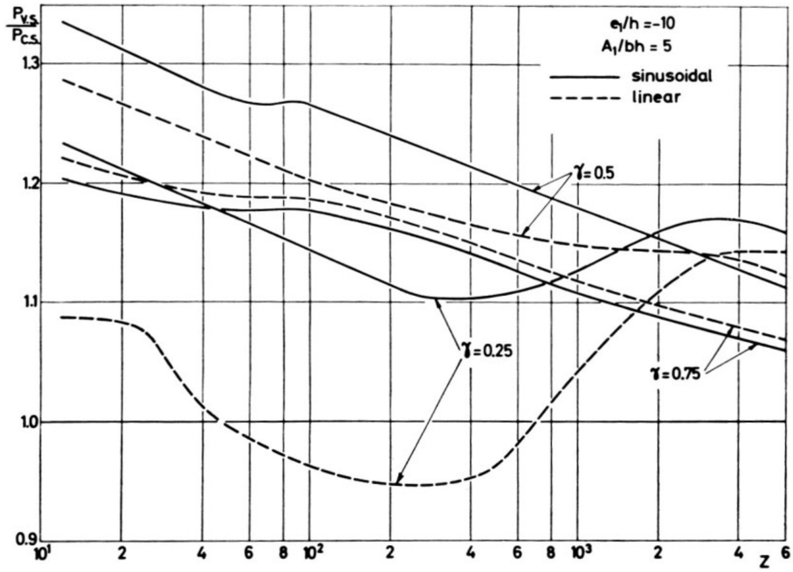

FIG. 11 — Comparison of structural efficiency of linear and sinusoidal height variation of stringers

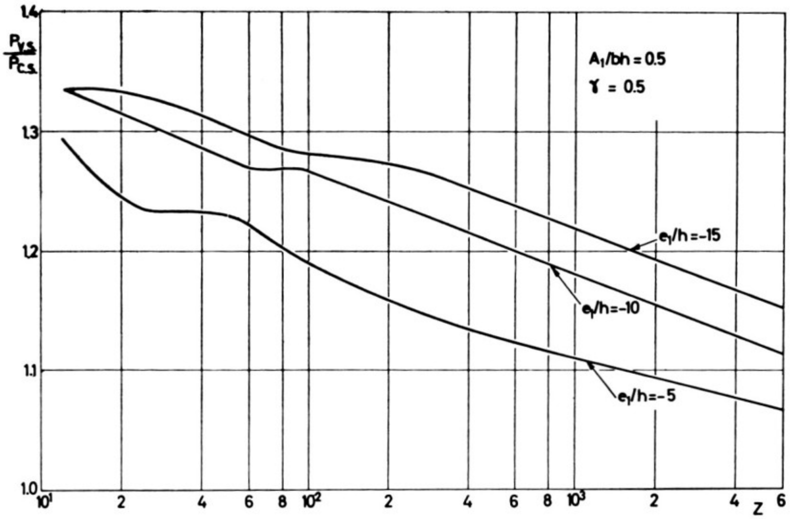

FIG. 12 — Effect of eccentricity on the relative structural efficiency of stringers with sinusoidal height variation

eccentricity is seen to increase the gain in structural efficiency possible with stiffeners of varying height. In practice, one has, however, also to check the section of maximum height for local buckling. Local buckling and other design considerations may obviously limit the feasible (e_1/h).

The gains in structural efficiency obtained with sinusoidal height variation in ring-stiffened cylindrical shells under lateral pressure are indicated in Table 1. Some typical shells with fairly heavy rings, equivalent to uniform rings of $(A_2/ah)=0{\cdot}5$, $(e_2/h)=\pm 5$, are considered. 30–70% gains in critical lateral pressure are found. Height variation of rings appears, therefore to be a promising approach to optimisation under lateral pressure. Width variation of rings is obviously less effective. Nevertheless, gains in critical pressure of 20–30% are obtained for typical shells with $\gamma=0$.

It should be noted, however, that weight savings for a given load are less spectacular than the gains in critical load for a given weight (*see* ref. 28) — a result encountered in any optimisation study.

TABLE 1

STIFFENED CYLINDRICAL SHELLS WITH RINGS OF SINUSOIDAL HEIGHT VARIATION

Position of rings	$\frac{L}{R}$	$\frac{R}{h}$	Z	γ	$\frac{P_{v.s.}}{P_{c.s.}}$
	0·5	50	11·9	0·50	1·34
	0·5	500	119	0·25	1·33
	0·5	500	119	0·50	1·30
in	1·0	1000	954	0·25	1·38
	3·0	1000	8580	0·44	1·47
	4·0	2000	30,500	0·25	1·67
out	4·0	2000	30,500	0·25	1·69

6. OPTIMISATION OF CONICAL SHELLS WITH NON-UNIFORMLY SPACED RINGS

In ring-stiffened conical shells under hydrostatic pressure the local conditions of the sub-shells differ, and hence unequal stiffener spacing may be more efficient. The optimum configurations of conical shells with uniformly and non-uniformly spaced rings of rectangular cross-section are therefore studied and compared.

Before one embarks on an optimisation study one should scrutinise the assumptions to be used. One of the commonly used assumptions in the analysis of the local instability in a ring-stiffened cylindrical or conical shell subjected to hydrostatic pressure appears then to be unjustified and hence warrants a detailed discussion.

In a stringer-stiffened cylindrical shell subjected to axial compression, the load is shared by stringers and skin, and the axial stress is the load divided by the total cross-sectional area of skin and stringers. When this stress reaches the critical stress of the curved panel between two stringers, usually considered simply-supported, local instability has occurred. The local buckling in the corresponding case of a ring-stiffened cylindrical shell under lateral or hydrostatic pressure does not represent an obvious extension of that in the axially loaded stringer-stiffened shell, due to the different manner of load application.

Consider first lateral pressure loading. If the rings are very stiff relative to the sub-shells they will practically not distort and the sub-shell behaves like a simply-supported cylindrical shell. The applied circumferential membrane stress is then $\sigma_\phi = (pR/h)$, where h is the thickness of the skin, and the shell prebuckling stress is not noticeably relieved by the stiffeners, as it was in the case of the axially loaded stringer-stiffened shell. The difference between the two cases becomes obvious if one imagines perfectly rigid stiffeners. No buckling is then possible in the axially compressed stringer-stiffened shell, provided rigid end rings transmit the load, whereas in the ring-stiffened shell under lateral pressure the buckling of the sub-shells is hardly affected, except for slight changes in the boundary conditions. These boundary effects, caused by increase in ring stiffness, consist of an effect on the prebuckling deformation investigated in 1932[30] and reconsidered recently in a more precise manner[31,32], and of a rotational restraint effect during buckling[31]. The prebuckling deformation effect increases the buckling pressure noticeably only in extremely short shells, whereas the rotational restraint during buckling may be appreciable even for sub-shells with Z up to 10.

Minimum-weight analyses[6,8] yield configurations with many closely spaced rings. The buckling behaviour of the resulting very short sub-shells approaches that of a long flat plate[33]. For lateral pressure the limiting case is a plate loaded by σ_ϕ and the corresponding plate factor $K=4$. Though the very small length of the sub-shells will augment the boundary effects, this increase will not be directly proportional to the area of the rings. Hence the assumption (used for example in ref. 5) that the applied stress for local buckling is $\sigma_\phi = (pR/\bar{h})$ where $\bar{h}$ is the equivalent thickness of the stiffened cylindrical shell, $\bar{h} = h[1+(A_2/ah)]$, does not appear justified for ring-stiffened cylindrical shells under lateral pressure. This assumption is even less justified for hydrostatic pressure loading. The buckling behaviour of very short sub-shells under hydrostatic pressure again approaches that of a long flat plate[33]. Now, however, the limiting case is a plate loaded in two perpendicular directions by σ_ϕ and $\sigma_x = (pR/2h)$. An analysis of such a plate shows that for a long plate the axial stress component becomes dominant. As the plate lengthens, a buckling pattern of a plate free at the short ends is approached, with the plate factor $K = 1$. The conclusion reached in ref. 31, that very short shells with $Z < 1{\cdot}89$ buckle axisymmetrically under hydrostatic pressure has essentially the same meaning. Only σ_x affects axisymmetric buckling (or Euler type buckling in the case of the long plate). Hence the rings cannot affect local buckling, except for some rotational boundary restraint, which again can only be very small with the rings of small torsional stiffness considered here. The assumption that the applied stress depends on the equivalent thickness for hydrostatic pressure loading used in refs. 5, 6, 7 and 8, is therefore not justified.

In conical shells under hydrostatic pressure, as in cylindrical shells, rings are the most efficient stiffeners, except in very short shells. A minimum-weight analysis and optimisation analysis (for fixed number of rings) is given in ref. 6 for uniformly spaced rings. Similar analyses for non-uniformly spaced rings are now derived.

For the very closely spaced rings demanded by minimum-weight designs, the sub-shell behaves as a simply-supported long plate and local buckling is determined by

$$(\sigma_x/E) = (pax\tan\alpha/2Eh) = [\pi^2h^2/12(1-\nu^2)a_\delta^2] \tag{11}$$

where a_δ is the length of a sub-shell. The ring spacing law that determines a_δ is

$$a_\delta = a_{0\delta}/x^\delta \tag{12}$$

It should be pointed out that $a_{0\delta}$, defined as the ring spacing when $x = 1$, is a mathematical parameter devoid of physical meaning, since there exists no sub-shell whose midpoint is $x = 1$. Substitution for a_δ in terms of $a_{0\delta}$, in eqn. (11),

$$(p/E) = [\pi^2h^3x^{(2\delta-1)}/6(1-\nu^2)a\tan\alpha a_{0\delta}^2] \tag{13}$$

and then

$$a_{0\delta} = \{\pi^2h^3k_1^{2\delta-1}/[6(1-\nu^2)a\tan\alpha(p/E)]\}^{1/2} \tag{14}$$

where for $\delta > 0{\cdot}5$, $k_1 = 1$ and for $\delta < 0{\cdot}5$, $k_1 = x_2$.

Since minimum-weight designs require many more rings than feasible in practice, more realistic optimal configurations can be obtained if the number of rings is specified as a practical restraint. The ring-spacing is then no longer small enough to ensure 'plate behaviour' and the sub-shells are short conical shells, considered simply-supported, whose buckling is determined[34] by

$$(p/E) = 0{\cdot}92(\bar{\rho}_{av}/a_\delta)(h/\bar{\rho}_{av})^{2\cdot5}g(\psi) \tag{15}$$

and due to shortness of the sub-shells $g(\psi) \simeq 1$. Hence

$$a_{0\delta} = \{0{\cdot}92h^{2\cdot5}k_2^{(\delta-1\cdot5)}/[(a\tan\alpha)^{1\cdot5}(p/E)]\} \qquad (16)$$

where for $\delta > 1{\cdot}5$, $k_2 = 1$ and for $\delta < 1{\cdot}5$, $k_2 = x_2$. It should be pointed out that eqns. (14) and (16) represent straight line conservative approximations to the actual curve of critical pressure against shell geometry, adopted for convenience of calculation in a manner similar to refs. 8 and 6.

For estimation of their local instability, the rings are represented by an infinite narrow plate simply-supported on one long side and free on the other, as in refs. 6 and 8, or, alternatively clamped on one long side and free on the other. The stress applied to the ring is computed with the assumption that rings and shell share the external load according to their cross-sectional area. For ring buckling to occur first, the shell must still be unbuckled and the skin will hence carry at least the part of the load proportional to its cross-sectional area. If the membrane stress distribution is unequal, due to wider ring spacing, the skin will carry a larger portion of the load and the rings a smaller load. The assumption of area-proportional load sharing is therefore at most conservative here. Hence

$$(\sigma_\phi/E)_R = [k_3\pi^2/12(1-\nu^2)](c/d)^2 = \{pax\tan\alpha/[Eh(1+dcx^\delta/a_{0\delta}h)]\} \qquad (17)$$

and then for $\nu = 0{\cdot}3$

$$(p/E)_R = 0{\cdot}904k_3(c/d)^2(h/a)[1+(dck_4^\delta/a_{0\delta}h)](1/k_4\tan\alpha) \qquad (18)$$

where $k_3 = 0{\cdot}5$ for simple-supports at one side, and $k_3 = 1{\cdot}33$ for one side clamped. k_4 can take any value between 1 and x_2. The correct value for k_4 is that that minimises $(p/E)_R$ in eqn. (18).

The general instability is computed with the approximate formula of ref. 12

$$(p/E)_G = 0{\cdot}92(\rho_{av}/l)(h/\rho_{av})^{2\cdot5}[(1+\eta_{2\delta}\bar{x}^\delta)^{0\cdot75} - (\rho_{av}/l)(h/\rho_{av})^{0\cdot5}\eta_{2\delta}\bar{x}^\delta]g(\psi)^{1-0\cdot3\delta} \qquad (19)$$

which neglects the eccentricity of the rings. Since the main aim of the present study is a comparison of the structural efficiency of non-uniformly spaced rings with that of uniformly spaced rings, and the eccentricity effects are approximately the same for both types of stiffening, the neglect of the eccentricity is not detrimental here.

The effective mean bending stiffness of the rings is represented by

$$\eta_{2\delta} = 0{\cdot}91(c/a_{0\delta})(d/h)^3 + \{3[(d/h)+1]^2/[\bar{x}^\delta + (1{\cdot}1a_{0\delta}h/cd)]\} \qquad (20)$$

and the equivalent thickness of the stiffened shell $\bar{h}$ (the thickness of an unstiffened conical shell of identical weight) is given by

$$\bar{h} = h\{1+(cd/a_{0\delta}h)[2(x_2^{(2+\delta)}-1)/(2+\delta)(x_2^2-1)]\} \qquad (21)$$

The investigation includes a minimum weight analysis as well as several optimisation studies with specified numbers of rings for uniform spacing. The calculations were performed in the following manner: A value for h, the shell wall thickness, is chosen and with eqns. (14) or (16) the required basic spacing $a_{0\delta}$ is computed for various ring-distribution factors δ. Then $\eta_{2\delta}$ is computed from eqn. (19) and the width of the ring c and its height d are found from eqns. (18) and (20). Finally the equivalent thickness of the stiffened shell is computed from eqn. (21).

The number of rings for non-uniform spacing can be found from the ring spacing law, eqn. (12). For hydrostatic pressure loading, comparison with uniformly spaced stiffening is based on the sub-shell with the largest mean radius of curvature. When the ring spacing varies according to eqn. (12), the length of this sub-shell is

$$(a_\delta)_m = a_{0\delta}/\{x_2 - [(a_\delta)_m/2]\}^\delta \tag{22}$$

Equation (12) can be expressed as a difference equation (*see* also Fig. 1)

$$a_\delta = a(\xi_{n+1} - \xi_n) = \{a_{0\delta}/[\tfrac{1}{2}(\xi_n + \xi_{n+1})]^\delta\} \tag{23}$$

where $a\xi_n$ is the distance along the generator from the vertex to the nth ring the boundary values of ξ are

$$\xi_0 = 1 \qquad \text{and} \qquad \xi_{N+1} = x_2 \tag{24}$$

and N is the total number of rings.

The number of rings for non-uniform spacing, or $a_{0\delta}$ for a given N, can be alternatively also calculated from a formula, obtained with a simple kinematic analogue (*see* also section 4 of ref. 26),

$$N = [a(x_2^{\delta+1} - 1)/a_{0\delta}(\delta+1)] - 1 \tag{25}$$

where N has to be rounded off to the nearest higher integer. The analogue is that of a body moving along the generator of the cone with a varying velocity. The velocity varies in such a manner that the body traverses the distance between any two rings, a_δ, in a constant time. The total time divided by the constant time in which a_δ is traversed gives the number of bays (or number of rings plus one).

In section 4 of ref. 26 results are presented for various geometries and loads. Here only results for one shell, subjected to a pressure $(p/E) = 1{\cdot}2 \times 10^{-6}$, and for three different ring distribution factors $\delta = 0$, 0·5, 1·5, are shown in Fig. 13. The equivalent thickness of the stiffened shell which represents the total weight is plotted against the number of rings N. A discontinuity in slope appears in all the curves of Fig. 13. This discontinuity is caused by transition from 'plate behaviour' of sub-shells to 'shell behaviour'. For very small ring-spacing (large number of rings) at the right of Fig. 13 'plate behaviour' is appropriate and eqn. (14) applies. As the ring-spacing increases towards the left of Fig. 13, the sub-shells have to be considered as conical shells and eqn. (16) applies. With increasing number of rings, or diminishing ring-

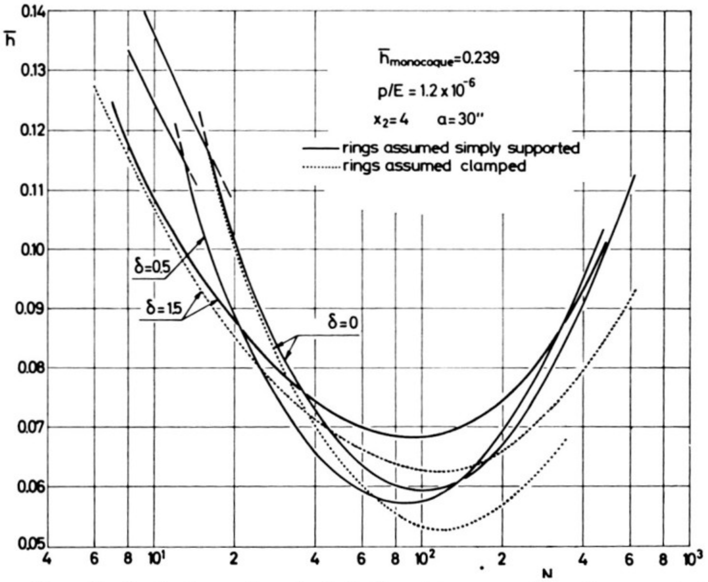

FIG. 13 — Optimisation of conical shells with non-uniformly spaced rings under hydrostatic pressure

spacing, the discontinuity is the point where the curves computed from eqns. (16) and (14) intersect, and to the right of which the approximate shell buckling formula, eqn. (16), is more conservative than the 'plate behaviour' approximation, which itself is slightly conservative. If the actual curve for the critical pressure of the sub-shells were used, no discontinuities would appear in Fig. 13.

One may note that for $\delta = 1{\cdot}5$ the transition occurs at an N beyond the minimum weight and is hence of no interest. For $\delta = 0$ and $\delta = 1{\cdot}5$ the computations were made with rings taken as a simply-supported – free plate ($k_3 = 0{\cdot}5$) and as a clamped – free plate ($k_3 = 1{\cdot}33$). Obviously the non-conservative clamped – free assumption yields smaller weights, but the differences are seen to be small, particularly in the practical range of N. Fig. 13 indicates that for minimum-weight design, $\delta = 0{\cdot}5$ results in the most efficient structure. This is not surprising since the minimum weight configuration has very small ring-spacing with corresponding 'plate behaviour'. In the 'plate regime' sub-shells of equal local stiffness are obtained with $\delta = 0{\cdot}5$, and hence this ring distribution is most efficient. The minimum-weight configurations, however, are not practical due to the large number of rings required, as mentioned earlier. In the optimal design region with a reasonable predetermined number of rings, $\delta = 1{\cdot}5$ results in a more efficient structure, since in the 'shell regime' $\delta = 1{\cdot}5$ yields sub-shells of equal local stiffness.

Whereas in the minimum-weight design region only small weight savings are possible with unequal ring-spacing (5–6%), considerable savings may be obtained in more practical configurations. For example, with $N=11$ the shell considered in Fig. 13 is 27% lighter with non-uniform ring-spacing ($\delta=1\cdot5$) than with uniform spacing ($\delta=0$). Or, if one aims at a reduction of manufacturing costs rather than weight saving, less rings are needed with varying ring-spacing. As an example, for $\bar{h}=0\cdot106$ inches in Fig. 13, 11 rings are needed with $\delta=1\cdot5$ whereas with $\delta=0$, 21 rings would be required.

Since the general instability pressure is calculated in this section with an approximate formula, eqn. (19), that neglects the eccentricity of the rings, the general instability pressure of some points in Fig. 13 has been recalculated with the more exact method of ref. 12. The differences in pressure are found to be small for inside rings, less than 6% in all cases, and only slightly larger for outside rings, 3–12%. One should also remember that a difference of say, 10% in critical pressure corresponds to a weight difference of less than 2%.

7. Structural Efficiency of Orthotropic Shells

Though in general orthotropic shells are less efficient than geometrically stiffened ones in their resistance to buckling, they may be optimal under certain conditions. Earlier studies of cylindrical shells, for example ref. 35, predicted considerable weight savings with orthotropic construction under external pressure and axial compression, especially for circumferential strengthening. More recent optimisation studies for orthotropic shells[36] show that under axial loads these predictions are not borne out and that, within the framework of linear theory, isotropic cylinders are more efficient in axial compression. The actual buckling loads realised in practice will, however, be larger in circumferentially stiffened orthotropic cylindrical shells than in corresponding isotropic ones, on account of the more stable post-buckling behaviour of circumferentially stiffened cylinders[24]. Since in glass fibre reinforced plastics the stiffness can be orientated in preferential directions without change in weight, orthotropic configurations may be advantageous in practice even under axial compression.

The weight savings predicted for cylindrical shells under lateral or hydrostatic pressure in ref. 35 have not been disputed. Recent theoretical work on orthotropic conical shells[37] (later confirmed experimentally[15]) also predicts noticeable weight savings under hydrostatic pressure. An example for commercially available glass fibre reinforced epoxy resins yielded there 9% weight savings with circumferential stiffening. In orthotropic conical shells under torsion[38] larger weight savings are possible and examples computed for commercially available glass fibre reinforced epoxy resins yielded weight savings of up to 15%.

One can extrapolate the conclusions arrived at earlier for ring-stiffened cylindrical shells with non-uniform rings, to orthotropic shells whose circumferential stiffness varies with x. Non-uniform orthotropy could be obtained in practice in fibrous composite shells and would yield noticeable weight savings.

8. Discrete Stiffeners

In the present paper, as in most general instability analyses of stiffened shells, the stiffeners are assumed to be distributed over the entire shell. On physical grounds this assumption appears reasonable for shells with many closely spaced stiffeners. In the case of ring-stiffened cylindrical shells under external pressure it has been shown to be valid even for small number of rings, except when there is only one central ring[39]; and for axisymmetric buckling under axial compression, the error in 'smearing' the rings was found less than 5%, except when the number of rings per longitudinal half wave was less than 2[40]. The large eccentricity effects found recently in stiffened shells and the desirability of variation of stiffener cross-section along the shell motivated a closer look at the discretely stiffened shell. Preliminary results obtained by one of our graduate students, R. Haftka, seem to confirm that for cylindrical shells under hydrostatic pressure, even with large eccentricities, discreteness of stiffeners is of practical importance only when their number is small. For a typical uniformly ring-stiffened cylindrical shell with $(L/R)=1{\cdot}0$, $(R/h)=2000$, $(A_2/ah)=0{\cdot}5$, $(I_{22}/ah^3)=5$ and $(e_2/h)=5$, the error introduced by 'smearing' the rings was found to be 16% for 2 rings; 7·5% for 3 rings and less than 0·05% when there are 4 or more rings.

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DISCUSSION

Prof. Ebner (University of Aachen, DVL, 51 Aachen, Templergraben 55): Zunächst möchte ich Herrn Prof. Singer und seinem Mitarbeiter zu der hervorragenden Arbeit beglückwünschen. Ich habe nun einige Fragen über die theoretischen Annahmen:

1. EXZENTRIZITÄT

In der vorgetragenen und in früheren Arbeiten der Verfasser wird der Einfluß exzentrischer Versteifungen nur durch die Abstände e_1 und e_2 der Neutralflächen für Biegung der Zylinderschale in Längs- und Umfangsrichtung von einer Bezugsfläche berücksichtigt. In einer deutschen Arbeit von Geier (WGL-Jahrbuch 1965, S. 440) werden unter der Annahme von elastischen Schichten aus den Koppelgliedern eines Elastizitätsgesetzes für die orthotrope Schale noch zwei weitere Exzentrizitäten e_{12} und e_3 abgeleitet welche die Abstände der Neutralflächen für die Schub- und Drillbeanspruchung darstellen sollen. Da mir die physikalische Bedeutung dieser Größen nicht klar ist, hätte ich gerne gewußt, ob die Verfasser des Vortrags hierfür eine Erklärung geben können und ob die Vernachlässigung dieser Größen in der vorliegenden Arbeit einen Einfluß hat?

2. VERSCHMIEREN

Die Grundlage für die vorliegende und andere theoretische Untersuchungen ist die Annahme, daß die Steifigkeiten der Ringe und Stringer auf die Schalenfläche 'verschmiert' werden dürfen. Bei dünner Haut und starken Versteifungen, besonders wenn diese in großen Abständen liegen, ist diese Annahme nicht mehr zulässig, weil die Haut dann vorher zwischen den Versteifungen ausbeult und eine Spannungsumlagerung stattfindet. Ich wäre dankbar, wenn der Vortragendeangeben könnte, bis zu welchen Grenzen eine 'Verschmierung' vorgenommen werden darf.

3. HISTORIE

Zum Schluß möchte ich noch darauf hinweisen, daß der Einfluß exzentrisch angeordneter Versteifungen auf die Beullast von Zylinderschalen bei achsialer und hydrostatischer Beanspruchung schon 1932 von Flügge in einer Arbeit in der Zeitschrift 'Ingenieurarchiv' untersucht wurde.

Prof. Singer and Mr. Baruch: The authors are grateful to Prof. Ebner for pointing out to them the early contribution of Prof. Flügge, which they had overlooked. In his 1932 paper, Prof. Flügge did indeed point out that the eccentricity of rings may be important. However, only one ring-stiffened shell was considered there, and the buckling mode with many longitudinal waves — that may reduce the critical load appreciably even for combinations of axial load and external pressure — was not taken into account there. Stringers, that are of equal importance, were not considered by Flügge in his calculations.

The recent work of Geier† mentioned by Professor Ebner differs from the earlier and present work of the authors in the assumed mathematical model. Geier replaces the stiffeners of the stiffened shell by a concentric orthotropic continuous layer, whereas the authors 'smear' the stiffeners to form a *cut* layer. For example, external rings are replaced in Geier's work by an orthotropic outer shell, whereas the authors replace them by a large number of parallel rings that cover the whole shell and touch each other, but are not connected to each other. The bending stiffness of Geier's layer and the smeared rings assumed by the authors is identical, but their torsional stiffness differs. Geier's model considerably over-estimates the torsional stiffness on account of the continuity of his layer that does not represent the rings correctly, whereas the model assumed by the authors slightly under-estimates the torsional stiffness. The authors feel, therefore, that their direct approach of stiffness distribution — computation of stiffnesses before 'smearing' them — is more realistic.

Earlier work by other investigators and the preliminary results obtained from the linear analysis with discrete stiffeners, indicate that within the framework of linear theory 'smearing' introduces only a negligible error, except when the number of stiffeners is very small. Experiments (*see* refs. 21, 22 and 26 of the paper) show that linear theory is adequate even for axial compression, provided the stiffeners are very closely spaced. With increase in stiffener spacing, however, the agreement between experiment and theory ceases to be satisfactory in this case (*see* ref. 15 of the paper and recent work by Katz at NASA‡). Hence it appears that 'smearing' is justified as long as linear theory is applicable. The investigation of discretely stiffened shells in

† GEIER, B., *Das Beulverhalten versteifter Zylinderschalen, Teil 1: Differentialgleichungen, Zeitschrift für Flugwissenschaften*, **14**, No. 7, pp. 306–323, July 1966.

‡ KATZ, L., 'Compression Tests on Integrally Stiffened Cylinders,' NASA TM X–53315, August 1965.

progress at the Technion aims at a more precise qualification of this statement.

As more experimental evidence becomes available, the bounds of validity of linear theory for stiffened shells will become clearer. At present the authors feel that these bounds cannot yet be specified with certainty.

A. van der Neut (Technological University, Delft, Kluyverweg 1, Delft, Netherlands): Mr. Chairman, First of all I should like to express my admiration for the thoroughness and extent of the work of the two authors.

In reply to Prof. Ebner's question whether post-buckling behaviour of skin panels can be taken into account I might make the comment that my early work dealt with just this case, since it was meant to apply to fuselage shells. For the same reason I considered at that time the favourable effect of outside eccentricity to be a useless by-product of my investigation, and the unfavourable effect of inside eccentricity to be the real product.

I have a question to put forward. In view of the possibility of the longitudinal wave length being a small multiple of the frame spacing, after having investigated the continuous structure ('smeared-out' stiffening), I studied the problem of *discrete* frames at constant spacing for the infinitely long cylinder. The differences with the continuous structure proved to be insignificant except for average half-wave lengths smaller than 1·5 or 2 times the frame spacing. However this problem contains an effect which puzzles me still: The tangential membrane stresses due to buckling put radial load upon the longitudinals and deflect them, thereby reducing the membrane stresses and consequently the effectivity of the skin in stabilising the structure. So far I have not been able to include this effect in the analytical solution for the *infinitely long* cylinder and I would like to know whether the authors have included it.

Prof. Singer and Mr. Baruch: The authors would like to thank Professor van der Neut for his comments. Professor van der Neut's work on eccentrically stiffened cylinders has been an encouragement to them in their efforts.

The effect of reduction of the membrane stresses due to the deflection of longitudinals caused by these membrane stresses is essentially a non-linear effect of the panel between discrete stiffeners. The effect is non-linear since the membrane stresses causing it will change only after a large deflection of the longitudinals has changed the geometry. Hence it is not included in the linear analysis presented.